Positive definite kernels on natural numbers

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Abstract

Notes for positive definite Kernels on natural numbers.

A Hilbert space interpretation of the equality $\frac{\zeta(s)^3}{\zeta(2s)} = \sum_{m,n\in\mathbf{N}} \frac{\gcd(m,n)^s}{m^s n^s}$

Prove that for $s > 2$ we have

$$\frac{\zeta(s)^3}{\zeta(2s)} = \sum_{m,n\in\mathbf{N}} \frac{\gcd(m,n)^s}{m^s n^s}$$

and ζ is the Riemann Zeta function.

I have a sketchy proof for this equality and would like to compare different solutions if possible.

Thanks for your help.

Here is the Hilbert space interpretation of this equality:

Consider the Jordan totient function for $s > 2$:

$$J_s(n) := n^s \prod_{p|n} (1 - \frac{1}{p^s})$$

Then we will use that:

$$\sum_{d|n} J_s(d) = n^s$$

Let e_n be the standard basis in the sequence space l_2 over the real numbers.

We define an embedding of the natural number n in this space via:

$$\phi_s(n) := \frac{1}{n^s} \sum_{d|n} \sqrt{J_s(d)} e_d$$

Then we have:

$$\langle \phi_s(a), \phi_s(b) \rangle = \frac{\gcd(a,b)^s}{a^s b^s}$$

Consider now the vector $\eta(s) := \sum_{n=1}^{\infty} \phi_s(n)$ in this Hilbert space.

It is then:

$$\eta(s) = \zeta(s) \sum_{d=1}^{\infty} \frac{\sqrt{J_s(d)}}{d^s} e_d$$

from which it follows that:

$$|\eta(s)|^2 = \zeta(s)^2 \sum_{d=1}^{\infty} \frac{J_s(d)}{d^{2s}} =^{\text{Wikip.}} \zeta(s)^2 \frac{\zeta(2s-s)}{\zeta(2s)} = \frac{\zeta(s)^3}{\zeta(2s)}$$

On the other hand we have:

$$|\eta(s)|^2 = \langle \eta(s), \eta(s) \rangle = \sum_{m,n} \langle \phi_s(m), \phi_s(n) \rangle = \sum_{m,n} \frac{\gcd(m,n)^s}{m^s n^s}$$

which proves the equality above using Hilbert spaces.

Question: Is this Hilbert space or positive definite kernel $k_s(a,b) = \frac{\gcd(a,b)^s}{a^s b^s}$ known in the literature and what are its usages?

Interpreting the Riemann zeta function as a squared norm of a Hilbert-Schmidt operator?

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Then we have:

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We can define a Hilbert-Schmidt operator R_s for $s > 2$ through:

$$R_s(e_n) := \phi_s(n)$$

and then we notice, that since $|\phi_s(n)|^2 = \frac{1}{n^s}$, we get:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} |\phi_s(n)|^2 =$$

$$= \sum_{n=1}^{\infty} |R_s(e_n)|^2 = |R_s|_{HS}^2$$

where $|.|_{HS}$ denotes the Hilbert-Schmidt norm. Hence $\sqrt{\zeta(s)}$ can be interpreted as the Hilbert-Schmidt norm $|R_s|_{HS}$. The definition for $x = \sum_{n=1} \langle x, e_n \rangle e_n$ is:

$$R_s(x) := \sum_{n=1} \langle x, e_n \rangle R_s(e_n) = \ldots = \sum_{n=1}^{\infty} D(x,n,s) \frac{\sqrt{J_s(n)}}{n^s} e_n$$

where we have set $D(x,n,s) := \sum_{m=1}^{\infty} \frac{\langle x, e_{mn} \rangle}{m^s}$, which is a Dirichlet series. We want to compute $|R_s(x)|^2$, which is given by another Dirichlet series:

$$\sum_{n=1} D(x,n,s)^2 \frac{J_s(n)}{n^{2s}} = |R_s(x)|^2 = \sum_{m,n} \frac{x_m x_n \gcd(m,n)^s}{m^s n^s}$$

where as usual $x_n := \langle x, e_n \rangle$.

Hilbert-Polya conjecture

This question comes out of my interest for positive definite kernels over the natural numbers. (I have collected some kernels with proofs).

First let me point to a connection between self-adjoint operators O on $H := l_2(\mathbb{N})$ and positive definite kernels k defined on the natural numbers.

Let $H := l_2(\mathbb{N})$ be the Hilbert space of sequences over the complex numbers. Suppose that we are given an operator $O = R^* R$, where R is a linear bounded operator. Then O is self adjoint, and denoting the standard orthonormal basis $(e_n)_{n \in \mathbb{N}}$ and $\phi(n) := R(e_n)$, we get:

$$\langle O(e_m), e_n \rangle = \langle R^* R(e_m), e_n \rangle = \langle R(e_m), R(e_n) \rangle = \langle \phi(m), \phi(n) \rangle =: k(m,n)$$

The left hand side denotes the entry of an infinte matrix $(\langle O(e_m), e_n \rangle) = (k(m,n))$. The right hand side denotes a positive definite kernel k on the natural numbers.

Probably the more interesting case is the other direction: Given a positive definite kernel k on the natural numbers (such as $k(a,b) = \gcd(a,b), \min(a,b), \frac{\min(a,b)}{\max(a,b)}, \frac{\gcd(a,b)}{ab}$), then by the Moore-Aronszajn theorem, there exists a feature mapping:

$$\phi : \mathbb{N} \to F$$

such that:

$$\langle \phi(a), \phi(b) \rangle_F = k(a,b)$$

Now define R through: $R(e_n) := \phi(n)$, and let $O = R^* R$. Then O is one self-adjoint operator associated with the positive definite kernel k over the natural numbers, and we have again the infinite matrix $(k(i,j))_{1 \leq i,j \leq \infty}$ corresponding to O.

I think this is the reason, why the positive definite kernels over the natural numbers seem to pop up everywhere, for example in [the formulation of the abc-conjecture][1]:

$$K(a,b) := \frac{2(a+b)}{\gcd(a,b) \operatorname{rad}\left(\frac{ab(a+b)}{\gcd(a,b)^3}\right)^2}$$

Another example where they show up, are [elliptic curves][2]:

Questions: 1. Suppose that O is one / the operator in the Hilbert-Polya conjecture. How does the Riemann hypothesis follow from the existence of O?

2. Is the spectrum of O (independent of the Hilbert-Polya conjecture) given by a limit in N where $\text{spec}(O_N), N \rightarrow \infty$, where $O_N = (k(m,n))_{1 \le m,n \le N}$ is a Gram matrix. In symbols:

$$\text{spec}(O) =^{?} \lim_{N->\infty} \text{spec}(O_N)$$

?

[1]: https://mathoverflow.net/questions/449483/abc-conjecture-and-positive-definite-kernels-again [2]: https://mathoverflow.net/questions/361894/discriminant-of-elliptic-curve-frey-hellegouarch-j-invariant-and-positive-def

Hilbert-Polya conjecture and Grommer inequalities

The Grommer inequalities are equivalent to RH and formulated [here on page 20][1]:

Let

$$\Xi(t) := \xi(1/2 + it)$$

Then RH is equivalent to : All zeros of $\Xi(t)$ are real. The functional equation for ζ is equivalent to $\Xi(t) = \Xi(-t)$. Let $Y(t) := \Xi(\sqrt{t})$ and let

$$-\frac{Y'(t)}{Y(t)} = s_1 + s_2 t + s_3 t^2 + \cdots$$

and set $k(a,b) := s_{a+b}$. Let $D_N := \det((k(m,n))_{1 \le m,n \le N} =: \det(K_N)$

Then the Grommer inequalities are equivalent to RH:

$$\text{RH} \iff D_N > 0 \forall N \in \mathbb{N}$$

But this is by the [Sylvester criterion][2] equivalent to the the matrices K_N being positive definite ($K_N > 0$) for all $N \in \mathbb{N}$.

What is unclear to me: Is $K_N > 0 \forall N \in \mathbb{N}$ equivalent to the function k being positive definite over the natural numbers?

If this is the case, one could have a candidate for the Hilbert-Polya conjecture: By the Moore-Aronszajn theorem, there exists a mapping:

$$\phi : \mathbb{N} \rightarrow H$$

in some Hilbert space H, such that:

$$k(a,b) = \langle \phi(a), \phi(b) \rangle \, \forall a, b \in \mathbb{N}$$

Define on this Hilbert space the operator R via:

$R(e_n) := \phi(n)$

where $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis in the Hilbert space.

Set $O = R^* R$. Then O is a self-adjoint operator, with the property:

$$\langle O(e_m), e_n \rangle = \langle R^* R(e_m), e_n \rangle = \langle R(e_m), R(e_n) \rangle = k(m,n)$$

Then by Grommer inequalities and Sylvester's criterion :

O is a positive operator with the property that $\langle O(e_m), e_n \rangle = k(m,n) \iff k$ is a positive definite function on the natural numbers $\iff$ RH.

[1]: https://aimath.org/ kaur/publications/90.pdf [2]: https://en.wikipedia.org/wiki/Sylvester

References